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OPTIMAL OPERATING MODES WITH CHATTERING SWITCHINGS IN MANIPULATOR CONTROL PROBLEMS†

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In the problem of an m-link manipulator control possessing a singular $2m$th-order mode it is shown that optimal trajectories attain a singular mode in a finite time with an infinite number of control switchings.

If an optimal control problem is affinely generated by a scalar control u, the Pontryagin function can be presented in the form $H = H_0 + uH_1$, where H_0 and H_1 are functions of the phase and adjoint variables. If $H_1 \neq 0$ along the trajectory, the control is uniquely determined as a function of time from the maximum condition. The control function is piecewise-constant and the corresponding trajectory is piecewise-smooth. A trajectory on which the control is not uniquely determined from the maximum condition is called a singular trajectory [1]. If a problem is affine in scalar control, then $H_1 \equiv 0$ along a singular trajectory. In order to determine the control on a singular trajectory, it is necessary to differentiate the identity $H_1 \equiv 0$. It is well known that a non-zero coefficient of control u can arise for the first time only at an even step of differentiation $2q$. The number q is referred to as the order of the singular trajectory. The necessary condition for optimality of a singular trajectory is the following Kelley's condition [2]

$$(-1)^q \frac{\partial}{\partial u} \frac{d^{2q} H_1}{dt^{2q}} \leq 0$$

The study of singular trajectories and their conjugations with non-singular trajectories served to develop the theory of chattering control. A chattering trajectory (ChT) is a trajectory with an infinite number of control switchings in a finite time interval. Chattering trajectories were proved to be typical in [3, 4], that is, it was proved that for an open set of Hamiltonian systems of Pontryagin's maximum principle with a scalar control a sub-manifold of finite codimensionality exists through each point of which a single-parameter family of ChTs passes.

One of the causes of chattering is the conjugation of a non-singular trajectory and a singular one of even order. It is well known [2] that if a singular trajectory of even order satisfies Kelley's condition in strict form

$$(-1)^q \frac{\partial}{\partial u} \frac{d^{2q} H_1}{dt^{2q}} < 0$$

then the conjugation of a piecewise-smooth non-singular trajectory with a singular one is non-optimal (the Kelley–Kopp–Moyer theorem). Therefore, if an optimal trajectory consists of a singular arc of even order and a non-singular arc, then the last one contains an infinite number of control switches. A complete theory of ChTs of the second order was constructed in [4, 5]. For problems with singular higher-order modes particular results have been obtained in [4].

Below we prove that for a certain class of optimal control problems with singular modes of any even order ChTs are optimal in the neighbourhood of the singular mode. We prove that in problems of controlling a multilink manipulator with elastic joints between links singular modes of high order are realized and optimal trajectories are ChTs.

1. SINGULAR MODES AND CHATTERING TRAJECTORIES

Consider the following problem

$$\int_0^\infty x_n^2(t)dt \to \inf \tag{1.1}$$

†*Prikl. Mat. Mekh.* Vol. 64, No. 1, pp. 19–28, 2000.

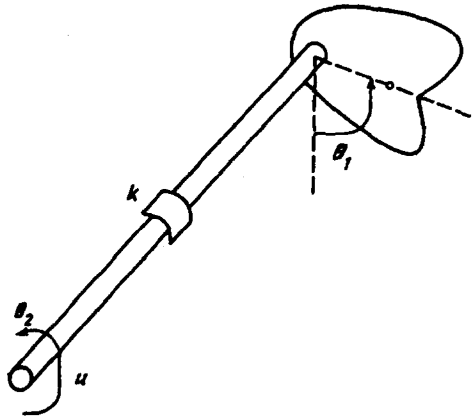

Fig. 1.

$$\dot{x}_1 = u + f_1(x), \quad \dot{x}_2 = x_1 + f_2(x), \quad \ldots, \quad \dot{x}_n = x_{n-1} + f_n(x) \tag{1.2}$$

Here $x = (x_1, \ldots, x_n) \in \mathbf{R}^n, f(x) = (f_1(x), \ldots, f_n(x)) \in C^\infty(\mathbf{R}^n, \mathbf{R}^n)$. The initial condition $x(0) = x^0$. The scalar control u is bounded: $u_- \leqslant u \leqslant u_+, u_- < 0, u_+ > 0$.

We investigate the behaviour of the solutions of problem (1.1), (1.2) in the vicinity of the origin which is a singular trajectory.

Let us consider the control system that is the principal part of system (1.2)

$$\dot{x}_1 = u, \quad \dot{x}_i = x_{i-1}, \quad 2 \leqslant i \leqslant n \tag{1.3}$$

It was proved in [4, 5] that in problem (1.1), (1.3) (i.e. when $f(x) \equiv 0$) optimal solutions reach the origin in a finite time. The upper bound of the time to reach the origin considered as function of an initial point was found. To obtain these results the homogeneity of the problem with respect to the action of the one-parameter group $G = \{g_\lambda, \lambda > 0\}$ of diffeomorphisms $\mathbf{R}^n$ was used in [4, 5]: $g_\lambda(x) = (\lambda x_1, \ldots, \lambda^n x_n)$. This group will be called a Fuller group.

We assume that the perturbation $f(x)$ is small with respect to the principal part of system (1.2). Namely, let $f(x)$ be small in the sense of the action of the Fuller group. This means that for a certain constant $C_0 > 0$ the following bounds

$$\overline{\lim_{\lambda \to +0}} \frac{|f_i(g_\lambda(x))|}{\lambda^i} < C_0, \quad i = 1, \ldots, n \tag{1.4}$$

hold uniformly in x from the set $\{x : |x_i| \leqslant 1, i = 1, \ldots, n\}$.

Problem (1.1), (1.2) is inhomogeneous with respect to the action of the Fuller group. However, it is possible to show that the qualitative behaviour of trajectories is the same as in problem (1.1), (1.3) (with $f(x) \equiv 0$).

For each $r > 0$ we define the set

$$Q_r = \{x \in \mathbf{R}^n : |x_i| \leqslant r^i, \; i = 1, \ldots, n\}$$

For problem (1.1), (1.2), (1.4) the following theorem [6] holds.†

Theorem 1. A number $r^\circ \geqslant 0$ exists such that for all $r \leqslant r^\circ$, the following statements hold.
1. The optimal trajectory $\hat{x}(t), \hat{x}(0) = x^\circ \in Q_r$, exists.
2. The optimal trajectory $\hat{x}(t)$ reaches the origin in a finite time not exceeding const r.

Remark. The time bound is uniform in all $x^\circ \in Q_r$ and the constant in the bound is independent of r.

The proof of Theorem 1 is based on the construction of a piecewise-smooth Lyapunov function for control system (1.2). This Lyapunov function is the same for all perturbations $f(x)$ satisfying condition (1.4).

†See also MANITA L. A., Asymptotic behaviour of extremals in the vicinity of singular trajectories of high order. Candidate dissertation, Moscow, 1996.

It is shown in [5] that for problem (1.1), (1.3) with $f(x) \equiv 0$ and even n optimal trajectories are ChTs. Using the results of Theorem 1 we will prove that the existence of chattering solutions is stable with respect to a certain class of perturbations.

Consider a perturbation $f(x)$ of the special form

$$f_i(x) = \sum_{k=i}^{n} \upsilon_{ik} x_k + h_i(x_n) \tag{1.5}$$

where $h_i(x_n) \in C^\infty(\mathbf{R})$ are non-linear functions that depend only on the variable x_n, and $h_i(0) = 0, h_i'(0) = 0$. The functions $f_i(x)$ $(i = 1, \ldots, n)$ of the form (1.5) are small with respect to the action of Fuller group (1.4).

Theorem 2 [6]. For problem (1.1), (1.2), (1.5) the following statements hold.

1. The origin $x \equiv 0$ is a singular trajectory of order n.
2. If n is even, then $\delta > 0$ exists such that the optimal solutions $\hat{x}(t), \hat{x}(x) = x^\circ \in Q_r, r \leqslant \delta$, reach the origin in finite time with infinite numbers of control switchings.

Proof. We use Pontryagin's maximum principle. We write the Pontryagin function

$$H = \psi_1(u + f_1(x)) + \sum_{i=2}^{n} \psi_i(x_{i-1} + f_i(x)) - x_n^2/2 = H_0 + uH_1, \quad u = \begin{cases} u_-, & \psi_1 < 0 \\ u_+, & \psi_1 > 0 \end{cases}$$

The system of adjoint equations has the form

$$\dot{\psi}_i = -\frac{\partial H}{\partial x_i} = -\psi_{i+1} - \sum_{k=1}^{i} \upsilon_{ki}\psi_k, \quad 1 \leqslant i \leqslant n-1$$

$$\dot{\psi}_n = -\frac{\partial H}{\partial x_n} = x_n - \sum_{k=1}^{n} \upsilon_{kn}\psi_k - \sum_{k=1}^{n} \psi_k h_k'(x_n)$$

Let us find singular trajectories, i.e. trajectories on which the coefficient of control u in the Pontryagin's function equals zero. We have

$$H_1 = \psi_1 \equiv 0 \Rightarrow \psi_2 \equiv 0 \Rightarrow \ldots \Rightarrow \psi_n \equiv 0 \Rightarrow x_n \equiv 0 \Rightarrow \ldots \Rightarrow x_1 \equiv 0$$

Thus, the origin is a singular trajectory. Let us calculate its order. We have

$$\frac{dH_1}{dt} = \dot{\psi}_1 = -\psi_2 - \upsilon_{11}\psi_1$$

$$\frac{d^2H_1}{dt^2} = -\dot{\psi}_2 - \upsilon_{11}\dot{\psi}_1 = \psi_3 + \sum_{k=1}^{2} \upsilon_{k2}\psi_k + \upsilon_{11}(\psi_2 + \upsilon_{11}\psi_1) \equiv \psi_3 + \sum_{k=1}^{2} \alpha_{k3}\psi_k$$

$$\vdots$$

$$\frac{d^{j-1}H_1}{dt^{j-1}} = \varepsilon_{j-1}\dot{\psi}_{j-1} - \sum_{k=1}^{j-2} \alpha_{k,j-1}\dot{\psi}_k = \varepsilon_{j-1}\left(-\psi_j - \sum_{k=1}^{j-1} \upsilon_{k,j-1}\psi_k\right) +$$

$$+ \sum_{k=1}^{j-2} \alpha_{k,j-1}\left(-\psi_{k+1} - \sum_{l=1}^{k} \upsilon_{lk}\psi_l\right) \equiv \varepsilon_j\psi_j + \sum_{m=1}^{j-1} \alpha_{mj}\psi_m$$

$$\vdots$$

$$\frac{d^nH_1}{dt^n} = \frac{d}{dt}\left(\varepsilon_n\psi_n + \sum_{k=1}^{n-1} \alpha_{kn}\psi_k\right) = \varepsilon_n\left(x_n - \sum_{k=1}^{n} \upsilon_{kn}\psi_k - \sum_{k=1}^{n} \psi_k h_k'(x_n)\right) + \sum_{k=1}^{n-1} \alpha_{kn}\dot{\psi}_k =$$

$$= \varepsilon_n x_n + \sum_{k=1}^{n} \alpha_{k,n+1}\psi_k - \varepsilon_n \sum_{k=1}^{n} \psi_k h_k'(x_n) = \varepsilon_n x_n + \varepsilon_{n+1} \sum_{k=1}^{n} \psi_k h_k'(x_n) + F_1(\psi_1, \ldots, \psi_n)$$

$$\frac{d^{n+1}H_1}{dt^{n+1}} = \varepsilon_n \dot{x}_n + \varepsilon_{n+1} \sum_{k=1}^{n} (\dot{\psi}_k h_k'(x_n) + \psi_k h_k''(x_n)\dot{x}_n) + \sum_{k=1}^{n} \frac{\partial F_1}{\partial \psi_k}\dot{\psi}_k =$$

$$= \varepsilon_n x_{n-1} + \varepsilon_{n+1} \sum_{k=1}^{n} \psi_k h_k''(x_n) x_{n-1} + F_2(\psi_1, \ldots, \psi_n, x_n)$$

$$\vdots$$

$$\frac{d^{n+j-1} H_1}{dt^{n+j-1}} = \varepsilon_n x_{n-j+1} + \varepsilon_{n+1} \sum_{k=1}^{n} \psi_k h_k''(x_n) x_{n-j+1} + F_j(\psi_1, \ldots, \psi_n, x_n, \ldots, x_{n-j+2})$$

$$\vdots$$

$$\frac{d^{2n-1} H_1}{dt^{2n-1}} = \varepsilon_n x_1 + \varepsilon_{n+1} \sum_{k=1}^{n} \psi_k h_k''(x_n) x_1 + F_n(\psi_1, \ldots, \psi_n, x_n, \ldots, x_2)$$

$$\frac{d^{2n} H_1}{dt^{2n}} = \varepsilon_n u + \varepsilon_{n+1} \sum_{k=1}^{n} \psi_k h_k''(x_n) \dot{u} + F_{n+1}(\psi_1, \ldots, \psi_n, x_n, \ldots, x_1)$$

where $\varepsilon_j = (-1)^{j+1}$ $(j \geqslant 2)$, and the numbers α_{mj} are determined by the recurrence formula

$$\alpha_{mj} = \varepsilon_j v_{m,j-1} - \alpha_{m-1,j-1} - \sum_{k=m}^{j-2} v_{mk}, \quad 2 \leqslant j \leqslant n, \quad 1 \leqslant m \leqslant j-1, \quad \alpha_{mj} = 0, \quad m \leqslant 0$$

The functions F_j $(j = 1, \ldots, n+1)$ depend only on the adjoint variables $\psi_1, \ldots, \psi_n$ and on the phase variables $x_n, \ldots, x_{n-j+2}$, and $F_j(0) = 0$ for any j.

The principal term in $d^{n+j-1}H_1/dt^{n+j-1}$ is

$$\varepsilon_n x_{n-j+1} + \varepsilon_{n+1} \sum_{k=1}^{n} \psi_k h_k''(x_n) x_{n-j+1}$$

Differentiation of this term gives a control more rapidly than differentiation of the function F_j. Therefore the functions F_j have no influence on the order of the singular trajectory and on Kelley's condition. Thus, a control appears for the first time at the $2n$th differentiation.

Hence, the origin is a singular trajectory of order n, and if n is even, then the singular trajectory satisfies Kelley's necessary condition for optimality in the strict form

$$(-1)^q \frac{\partial}{\partial u} \frac{d^{2q} H_1}{dt^{2q}} = (-1)^n \varepsilon_n = -1 < 0$$

Problem (1.1), (1.2), (1.5) satisfies the conditions of Theorem 1. According to Theorem 1, the optimal trajectory $x(t)$ emerging from $x^\circ \in Q_r$, where r is sufficiently small, reaches the origin in a finite time

$$t_1 = t_1(x^\circ) \leqslant \text{const}\, r$$

and then remains at zero. The time estimate is uniform in $x^\circ \in Q_r$ and the constant in the estimate is independent of r. Thus $x(t) = 0$ when $t \geqslant t_1$. Then the control $u(t) = 0$ when $t \geqslant t_1$, and from Pontryagin's maximum principle it follows that $\psi_1(t) = 0$ when $t \geqslant t_1$. From the system of adjoint equations we obtain $\psi_i(t) = 0$ $(i = 2, \ldots, n)$ when $t \geqslant t_1$. Consequently, the optimal trajectory reaches the singular trajectory in a finite time t_1.

If n is even, then the singular trajectory of problem (1.1), (1.2), (1.5) satisfies all conditions of the Kelley–Kopp–Moyer theorem. Hence, the optimal control has an infinite number of switchings in a finite time interval. Theorem 2 is proved.

2. THE PROBLEM OF THE CONTROL OF A MULTILINK MANIPULATOR WITH ELASTIC JOINTS BETWEEN ITS LINKS

Investigations of the problems of manipulator control (including multilink manipulators) using methods from different areas of science complement each other and provide a more complete picture of manipulator operation [7–9].†

†See also AKULENKO, L. D., BOLOTNIK, N. N. and KAPLUNOV, A. A., Optimization of the control of manipulation robots. Preprint No.218, Institute of Problem Mechanics, Russian Academy of Sciences, Moscow, 1983, 72 pp.

BOISSONNAT, J. D., DEVILLERS, O., PREPARATA, F. P. and DONATI, L., Motion planning of legged robots: the spider robot problem. Research Report No. 1767, INRIA, France, 1992.

Optimal control methods enable new effects to be revealed that must be taken into account when solving specific application problems. Problems of robot manipulator control with second-order singular modes were considered in [8, 9].

For the time-optimal problem for a two-link manipulator a complete synthesis containing ChTs was constructed in [8]. It was shown† that, in the time-optimal problem for the robot-machine, the optimal trajectories are ChTs.

In the problems of manipulator control considered below singular modes of any even order are realized, and optimal trajectories attain a singular mode with chattering.

A system with a single elastic element. Consider a two-link manipulator. The links are joined by a spring of stiffness k. The first link is rigidly connected to a robot arm. A rotational force u, $-1 \leqslant u \leqslant 1$, is applied to the end of the second link (Fig. 1). This mechanical system is described by the following system of differential equations

$$J_1\ddot{\theta}_1 = -MgL\sin\theta_1 - k(\theta_1-\theta_2), \quad J_2\ddot{\theta}_2 = k(\theta_1-\theta_2)+u(t) \tag{2.1}$$

The scalar control u satisfies the constraint $|u| \leqslant 1$. The initial conditions are

$$\theta_1(0)=\theta_{10}, \quad \dot{\theta}_1(0)=\theta'_{10}, \quad \theta_2(0)=\theta_{20}, \quad \dot{\theta}_2(0)=\theta'_{20}$$

The problem consists in minimizing the following functional

$$\int_0^\infty (\theta_1-\gamma)^2 dt \to \inf \tag{2.2}$$

We put

$$\mathcal{X} = \left(\gamma, \ 0, \ \gamma+\frac{MgL}{k}\sin\gamma, \ 0\right)$$

Theorem 3. Let $|MgL \sin\gamma| < 1$. Then for problem (2.1), (2.2) the following statements hold.

1. $(\theta_1(t), \theta_1(t), \theta_2(t), \theta_2(t)) \equiv \mathcal{X}$ is a singular fourth-order trajectory.

2. For all initial positions $\theta^\circ = (\theta_1(0), \theta_1(0), \theta_2(0), \theta_2(0))$, sufficiently close to the point $\mathcal{X}$, the optimal control moves the robot arm to position $\theta_1 = \gamma$ in a finite time $t^\circ = t^\circ(\theta_1)$ with an infinite number of switchings points in $(0, t^\circ)$.

Proof. Put

$$a_1 = -\frac{MgL}{J_1}, \quad a_2 = -\frac{k}{J_1}, \quad a_3 = \frac{k}{J_2}, \quad a_4 = \frac{1}{J_2}$$

$$x_4 = \theta_1-\gamma, \quad x_3 = \dot{\theta}_1, \quad x_2 = -a_2(\theta_2-\gamma)+a_1\sin\gamma, \quad x_1 = -a_2\dot{\theta}_2$$

We define the new control

$$\tilde{u} = -a_2a_4u + a_1a_3\sin\gamma$$

Put $\alpha_\pm = k(1 \mp MgL \sin\gamma)/(J_1J_2)$. If $u = 1$, then $\tilde{u} = \alpha_+$; if $u = -1$, then $\tilde{u} = -\alpha_-$. From the assumption $|MgL \sin\gamma| < 1$ it follows that $\alpha_\pm > 0$. Problem (2.1), (2.2) can be written in the following form

$$\int_0^\infty x_4^2(t)dt \to \inf \tag{2.3}$$

†See BOISSONNAT, J.-D, CEREZO, A. and LEBLOND, J., A note on shortest paths in the plane subject to a constraint on the derivative of the curvature. Research Report No. 2160. INRIA, France, 1994.

DEGTIARIOVA-KOSTOVA, E. and KOSTOV, V., Irregularity of optimal trajectories in a control problem for a car-like robot. Research Report No. 3411. INRIA, France, 1998.

$$\dot{x}_1 = \tilde{u} - a_2 a_3 x_4 - a_3 x_2, \quad \dot{x}_2 = x_1$$
$$\dot{x}_3 = x_2 + x_4(a_1 \cos\gamma + a_2) + a_1 A(x_4)\cos\gamma + a_1 B(x_4)\sin\gamma, \quad \dot{x}_4 = x_3 \tag{2.4}$$

where

$$A(x_4) = \sin x_4 - x_4 = O(x_4^3), \quad B(x_4) = \cos x_4 - 1 = O(x_4^2)$$

The control u satisfies the constraint $-\alpha \leqslant u \leqslant \alpha_+$. The initial conditions have the form

$$x_1(0) = -a_2\theta'_{20}, \quad x_2(0) = -a_2(\theta_{20} - \gamma) + a_1 \sin\gamma$$
$$x_3(0) = \theta'_{10}, \quad x_4(0) = \theta_{10} - \gamma \tag{2.5}$$

We will show that problem (2.3), (2.4) satisfies the conditions of Theorem 2. Put

$$f_1(x) = -a_3 x_2 - a_2 a_3 x_4$$
$$f_3(x) = x_4(a_1 \cos\gamma + a_2) + a_1 A(x_4)\cos\gamma + a_1 B(x_4)\sin\gamma$$

Then

$$\dot{x}_1 = \tilde{u} + f_1(x), \quad \dot{x}_2 = x_1, \quad \dot{x}_3 = x_2 + f_3(x), \quad \dot{x}_4 = x_3$$

The functions $f_1(x)$ and $f_3(x)$ belong to class (1.5) considered in Theorem 2. In fact, put $v_{12} = -a_3$, $v_{14} = -a_2 a_3$, $v_{34} = -a_1\cos\gamma$, the other v_{ik} are equal to zero, and

$$h_3(x_4) = a_1 A(x_4)\cos\gamma + a_1 B(x_4)\sin\gamma, \quad h_i \equiv 0, \quad i = 1, 2, 4$$

From the definition of the functions $A(x_4)$ and $B(x_4)$ it follows that $h_3(0) = h'_3(0)$.

Therefore, Theorem 2 can be applied to problem (2.3), (2.4). Consequently, the origin is a singular fourth-order trajectory. From Theorem 2 it follows that, for initial conditions (2.5) from a sufficiently small neighbourhood of the origin, the optimal solutions reach the origin in a finite time t^0 with an infinite number of control switches in $(0, t^0)$.

Since problems (2.1), (2.2) and (2.3), (2.4) are equivalent, the above proof means, in terms of problem (2.1), (2.2), that $(\theta_1(t), \dot{\theta}_1(t), \theta_2(t), \dot{\theta}_2(t)) \equiv \mathcal{H}$ is a singular fourth-order trajectory. The optimal trajectories of problem (2.1), (2.2) reach the singular trajectory with an infinite number of control switches in a finite time interval. Thus, if initial state θ^0 is sufficiently close to the point $\mathcal{H}$ then to minimize functional (2.2) it is necessary to move the robot arm to the position

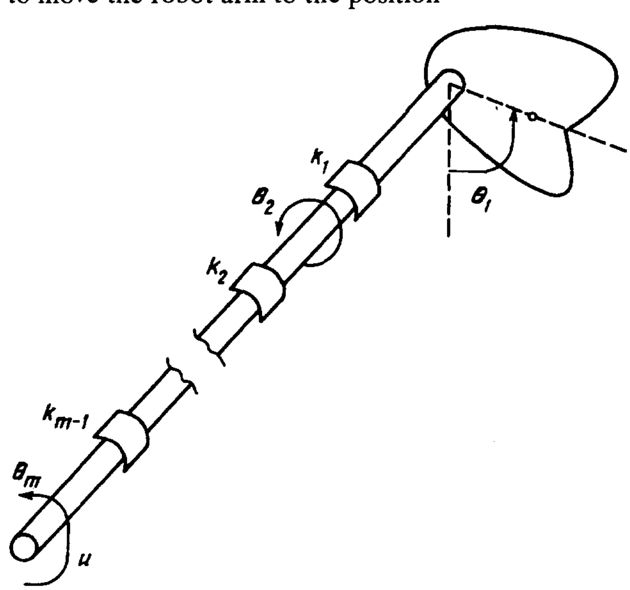

Fig. 2.

$$\theta_1 = \gamma, \quad \theta_2 = \gamma + \frac{MgL}{k}\sin\gamma$$

in a finite time with an infinite number of switchings of the force u. Theorem 3 is proved.

A system with several elastic elements. Based on manipulator model (2.1), (2.2), it is possible to propose examples of controlled mechanical systems possessing singular trajectories of any even order, which are reached by optimal solutions in a finite time with an infinite number of control switchings.

Consider a multilink manipulator. A manipulator consists of m links successively joined by springs of stiffnesses $k_1, \ldots, k_{m-1}$. The first link is rigidly connected to the robot arm. A rotational force u, $-1 \leqslant u \leqslant 1$, is applied to the mth link (Fig. 2). This mechanical system is described by the following system of differential equations

$$\begin{aligned}
J_1\ddot{\theta}_1 &= -MgL\sin\theta_1 - k_1(\theta_1 - \theta_2)\\
J_2\ddot{\theta}_2 &= -k_1(\theta_2 - \theta_1) - k_2(\theta_2 - \theta_3)\\
&\ldots\\
J_i\ddot{\theta}_i &= -k_{i-1}(\theta_i - \theta_{i-1}) - k_i(\theta_i - \theta_{i+1})\\
&\ldots\\
J_m\ddot{\theta}_m &= -k_{m-1}(\theta_m - \theta_{m-1}) + u(t)
\end{aligned} \tag{2.6}$$

The scalar control u satisfies the constraint $|u| \leqslant 1$. The initial conditions are

$$\theta_i(0) = \theta_{i0}, \quad \dot{\theta}_i(0) = \theta'_{i0}, \quad i = 1, \ldots, m$$

As in the case when $m = 2$, the problem consists of minimizing the following functional

$$\int_0^\infty (\theta_1 - \gamma)^2 dt \to \inf \tag{2.7}$$

We put

$$\mathcal{H}_m = \left(\gamma,\ 0,\ \gamma + \frac{MgL}{k_1}\sin\gamma,\ 0, \ldots, \gamma + MgL\sin\gamma\sum_{i=1}^{m-1}(k_i)^{-1},\ 0\right)$$

The main result for problem (2.6), (2.7) is as follows.

Theorem 4. Let $|MgL \sin\gamma| < 1$. Then the following statements hold.

1. $(\theta_1(t), \theta_1(t), \ldots, \theta_m(t), \theta_m(t)) \equiv \mathcal{H}_m$ is a singular trajectory of order $2m$.
2. For all initial positions $\theta^\circ = (\theta_1(0), \theta_1(0), \ldots, \theta_m(0), \theta_m(0))$ sufficiently close to the point $\mathcal{H}_m$, the optimal trajectories reach position $\mathcal{H}_m$ in a finite time with an infinite number of control switchings.

Proof. The proof is similar to the analysis in the case of two-link robot. It is based on the reduction of the problem to a form that satisfies the conditions of Theorem 2. We put

$$x_1 = \theta_1 - \gamma, \quad x_2 = \dot{\theta}_1$$

$$x_{2l+1} = \left(\theta_{l+1} - \gamma - MgL\sin\gamma\sum_{i=1}^{l}\frac{1}{k_i}\right)\prod_{i=1}^{l}\frac{k_i}{J_i}$$

$$x_{2l+2} = \dot{\theta}_{l+1}\prod_{i=1}^{l}\frac{k_i}{J_i}, \quad 1 \leqslant l \leqslant m-1$$

The point $\mathcal{H}_m$ corresponds to the origin in $(x_1, x_2, \ldots, x_{2m})$ coordinates. We define a new control

$$\tilde{u} = \frac{k_1 \ldots k_{m-1}}{J_1 \ldots J_m}(u - MgL\sin\gamma)$$

We put

$$f_2(x) = -\frac{MgL\cos\gamma}{J_1}\sin x_1 + \frac{MgL\sin\gamma}{J_1}(1-\cos x_1)$$

$$f_{2l}(x) = \frac{k_{l-1}^2}{J_{l-1}J_l}x_{2l-3} - \frac{k_{l-1}+k_l}{J_l}x_{2l-1}, \quad 2 \leq l \leq m-1$$

$$f_{2m}(x) = \frac{k_{m-1}^2}{J_{m-1}J_m}x_{2m-3} - \frac{k_{m-1}}{J_m}x_{2m-1}$$

$$f_{2l+1}(x) = 0, \quad 0 \leq l \leq m-1$$

We write the system in $(x_1, x_2, \dots, x_{2m})$ coordinates

$$\dot{x}_{2l-1} = x_{2l}, \quad \dot{x}_{2l} = x_{2l+1} + f_{2l}(x), \quad 1 \leq l \leq m-1$$

$$\dot{x}_{2m-1} = x_{2m}, \quad \dot{x}_{2m} = \tilde{u} + f_{2m}(x)$$

We reverse the numbering of the indices, i.e. we put

$$z_l = x_{2m-l+1}, \quad \tilde{f}_l(z) = f_{2m-l+1}(x), \quad 1 \leq l \leq 2m$$

Problem (2.6), (2.7) in $(z_1, z_2, \dots, z_{2m})$ coordinates has the form

$$\int_0^\infty z_{2m}^2(t)dt \to \inf$$
$$\dot{z}_1 = \tilde{u} + \tilde{f}_1(z), \quad \dot{z}_2 = z_1, \quad \dot{z}_{2l-1} = z_{2l-2} + \tilde{f}_{2l-1}(z), \quad \dot{z}_{2l} = z_{2l-1}, \quad 2 \leq l \leq m \tag{2.8}$$

The perturbation $\tilde{f}(z) = (\tilde{f}_1(z), \tilde{f}_2(z), \dots, \tilde{f}_{2m}(z))$ belongs to class (1.5). Therefore, Theorem 2 can be applied to problem (2.8). For problem (2.6), (2.7) it means that the point $\mathcal{K}_m$ is a singular trajectory of order $2m$. And optimal trajectories attain the singular trajectory with an infinite number of control switchings in a finite time.

Thus, if the initial state θ^0 is sufficiently close to the point $\mathcal{K}_m$, then to minimize functional (2.7) it is necessary to move the robot arm to the position

$$\theta_1 = \gamma, \quad \theta_2 = \gamma + \frac{MgL}{k_1}\sin\gamma, \quad \dots, \quad \theta_m = \gamma + MgL\sin\gamma\sum_{i=1}^{m-1} k_i^{-1}$$

in a finite time with an infinite number of switchings of the force u. Theorem 4 is proved.

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